

# PROPERLY EMBEDDED AREA MINIMIZING SURFACES IN HYPERBOLIC THREE SPACE

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## 1. INTRODUCTION

The construction of new examples of complete minimal surfaces in hyperbolic space has had a very powerful tool: the solvability of the asymptotic Plateau problem. The asymptotic Plateau problem in hyperbolic space basically asks the existence of an area-minimizing submanifold in  $\mathbb{H}^{n+1}$  which is asymptotic to a given submanifold  $\Gamma^{n-1} \subset \partial_\infty \mathbb{H}^{n+1}$ , where  $\partial_\infty \mathbb{H}^{n+1}$  represents the sphere of infinity of  $\mathbb{H}^{n+1}$ , which we also call *the ideal boundary of hyperbolic space*.

Using methods from the Geometric Measure Theory, Michael Anderson [1] solved the asymptotic Plateau problem for absolutely area-minimizing submanifolds in any dimension and codimension.

Anderson did not impose any restriction to the topology the solutions he gets, so we cannot get any idea about their topological properties. In this way, it becomes interesting (as in the classical Plateau problem) to find the area-minimizing solution but fixing *a priori* the topological type. In [2], Anderson focused on the asymptotic Plateau problem with the type of a disk and provided an existence result in dimension 3.

Moreover, in [2], Anderson built a special Jordan curves in  $\partial_\infty \mathbb{H}^3$ , such that the surface obtained as a solution to the asymptotic Dirichlet problem cannot be a plane. In fact, he built examples of genus  $g > g_0$  for a particular genus  $g_0$ . In the same context, de Oliveira and Soret [7] demonstrated the existence of complete and stable minimal surfaces in hyperbolic 3-space for any orientable *finite* topological type<sup>1</sup>. They also studied the isotopy type of these surfaces in some special cases. The main difference with the result of Anderson is that Anderson begins with asymptotic data, and gives an area-minimizing surface with that particular data but without any kind of control over the topological type, while Oliveira and Soret start with a surface with boundary and build a stable embedded minimal surface in the hyperbolic space whose asymptotic (or ideal) boundary is determined essentially by the surface. In this setting, we can frame the following conjecture:

**Conjecture** (A. Ros). Every open orientable surface<sup>2</sup> can be properly and minimally embedded in  $\mathbb{H}^3$ .

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<sup>1</sup>A surface has finite topological type if it has the topology of a compact surface minus a **finite** number of points.

<sup>2</sup>We say that a surface is open if it is not compact and without boundary.

This paper is devoted to give a positive answer to the problem above. To be more precise we prove:

**Theorem A.** *Any open orientable surface can be properly embedded in  $\mathbb{H}^3$  as an area minimizing surface. Moreover, the above embedding can be constructed in such a way that the limit sets of different ends are disjoint.*

The definition of “area-minimizing” and “uniquely area-minimizing” surfaces can be found in Section 3 (Definition 3.1.) The fundamental tool in solving this problem has been the bridge principle at infinity (Section 3) which can be stated in these terms:

**Theorem B (Bridge principle at infinity).** *Let  $S$  be a properly embedded, uniquely area-minimizing surface with finite topology in  $\mathbb{H}^3$  that extends  $C^\infty$  to an embedding into  $\overline{\mathbb{H}^3}$ . Let  $\Gamma$  be a smooth arc in  $\partial_\infty \mathbb{H}^3$  meeting  $\partial_\infty S$  orthogonally and satisfying  $\Gamma \cap \partial_\infty S = \partial \Gamma$ .*

*Consider a sequence of bridges  $P_n$  on  $\partial_\infty \mathbb{H}^3$  that shrink nicely to  $\Gamma$ . If  $S$  is strictly  $L^\infty$ -stable (see Definition 3.4), then for all large enough  $n$ , there exists a strictly  $L^\infty$ -stable, uniquely area minimizing surface  $S_n$  which is properly embedded in  $\mathbb{H}^3$  and satisfying:*

- 1)  $S_n$  extends  $C^\infty$  to  $\overline{\mathbb{H}^3}$ ;
- 2)  $\partial_\infty S_n = (\partial_\infty S \setminus \partial P_n) \cup (\partial P_n \setminus \partial_\infty S)$ ;
- 3) The sequence  $S_n$  smoothly converges to  $S$  on compact subsets of  $\overline{\mathbb{H}^3} \setminus \Gamma$ .
- 4) The surface  $\overline{S_n}$  is homeomorphic to  $\overline{S} \cup P_n$ .

This bridge principle gives us some flexibility in order to construct properly embedded area-minimizing surfaces in  $\mathbb{H}^3$  with arbitrary infinite topology and some kind of regularity at infinity.

**Theorem C.** *If  $S$  is an open orientable surface with infinite topology, then there exists a proper area-minimizing embedding of  $S$  into  $\mathbb{H}^3$  such that the limit set in  $\partial_\infty \mathbb{H}^3$  is a smooth curve except for one point.*

Finally, we would like to point out that the same methods allow us to construct properly embedded area-minimizing surfaces so that the limit set is the whole ideal boundary  $\partial_\infty \mathbb{H}^3$ .

## 2. PRELIMINARIES

Throughout this paper  $\mathbb{H}^{n+1}$  will represent the  $(n+1)$ -dimensional hyperbolic space. We will use the models:

- (1) **Poincaré’s ball model:** the open unit ball  $\mathbb{B}^{n+1}$  of  $\mathbb{R}^{n+1}$  endowed with

$$\text{Poincaré’s metric } ds^2 := 4 \frac{\sum_{i=1}^{n+1} dx_i^2}{(1 - \sum_{i=1}^{n+1} x_i^2)^2}.$$

- (2) **Poincaré’s half-space model:** the upper half-space  $\{x_{n+1} > 0\} \subset \mathbb{R}^{n+1}$ ,

$$\text{endowed with the metric } ds^2 := \frac{1}{x_{n+1}^2} \sum_{i=1}^{n+1} dx_i^2.$$

Let  $\overline{\mathbb{H}}^{n+1}$  denote the compactification of  $\mathbb{H}^{n+1}$ . As we mentioned in the introduction, we shall denote the ideal boundary as  $\partial_\infty \mathbb{H}^{n+1} := \overline{\mathbb{H}}^{n+1} \setminus \mathbb{H}^{n+1}$ . Observe that  $\partial_\infty \mathbb{H}^{n+1}$  is diffeomorphic to the sphere  $S^n$ .

**2.1. Simple exhaustions.** One of the main tools in the proofs of the theorems stated in the introduction is the existence of a particular kind of exhaustions for any open surface. In [3], Ferrer, Meeks and the first author proved that any open orientable surface  $S$  of infinite topology has a smooth compact exhaustion  $S_1 \subset S_2 \subset \cdots S_n \subset \cdots$ , called a *simple exhaustion*. The defining properties for this exhaustion to be simple when  $S$  is *orientable* are:

- (1)  $S_1$  is a disk.
- (2) For all  $n \in \mathbb{N}$ , each component of  $S_{n+1} - \text{Int}(S_n)$  has one boundary component in  $\partial S_n$  and at least one boundary component in  $\partial S_{n+1}$ .
- (3) For all  $n \in \mathbb{N}$ ,  $S_{n+1} - \text{Int}(S_n)$  contains a unique nonannular component which topologically is a pair of pants or an annulus with a handle.

If  $S$  has finite topology (with genus  $g$  and  $k$  ends), then we say the compact exhaustion is *simple* if properties 1 and 2 hold, property 3 holds for  $n \leq g + k$ , and when  $n > g + k$ , all of the components of  $S_{n+1} - \text{Int}(S_n)$  are annular.

**2.2. Limit sets.** We are also interested in the asymptotic behavior of the minimal surfaces we are going to construct. So, we need some background about the limit set of an end.

**Definition 2.1.** Let  $\psi: S \rightarrow \mathbb{H}^3$  be a proper embedding of a surface  $S$  with possibly non-empty boundary. The **limit set** of  $S$  is  $L(S) = \bigcap_{\alpha \in I} (\psi(S) - \psi(C_\alpha))$ , where  $\{C_\alpha\}_{\alpha \in I}$  is the collection of compact subdomains of  $S$  and the closure  $\overline{\psi(S) - \psi(C_\alpha)}$  is taken in  $\overline{\mathbb{H}^3}$ . The **limit set  $L(E)$  of an end  $E$  of  $S$**  is defined to be the intersection of the limit sets of all properly embedded subdomains of  $S$  with compact boundary which represent  $E$ . Notice that  $L(S)$  and  $L(E)$  are closed sets of  $\partial_\infty(\mathbb{H}^3)$ .

### 3. BRIDGE PRINCIPLE AT INFINITY

The other tool in the construction of our minimal embeddings with arbitrary topology is a sort of bridge principle at infinity for properly embedded area-minimizing surfaces in  $\mathbb{H}^3$ .

**Definition 3.1.** Given a properly embedded surface  $S$  in  $\mathbb{H}^3$  we say that  $S$  is **area minimizing** if any compact piece is area minimizing among all the surfaces with the same boundary. We will say that  $S$  is **uniquely area minimizing** if it is the **only** area minimizing surface with its ideal boundary.

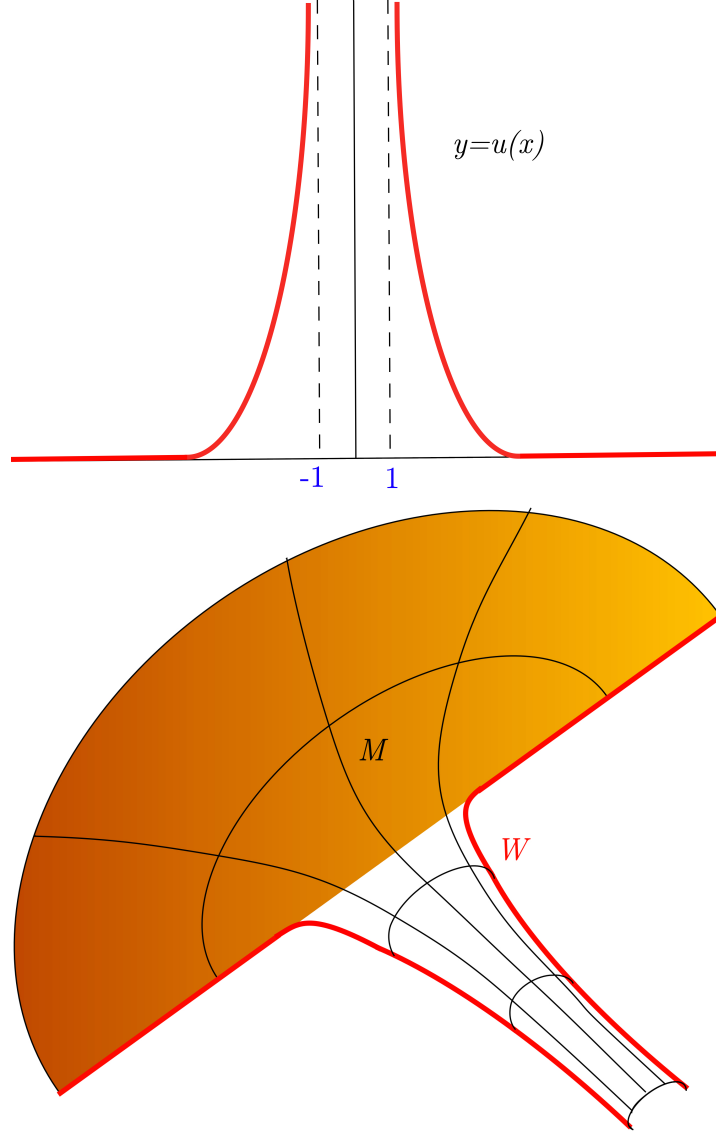
**Definition 3.2.** Suppose  $S \subset \mathbb{H}^3$  is a (possibly nonorientable) surface with un-oriented boundary.  $S$  is **area-minimizing mod 2** if  $S$  has least area amongst all surfaces (orientable or nonorientable) with the same boundary. If  $S$  is complete we say  $S$  is **minimizing mod 2** if each compact is area-minimizing mod 2.

#### 3.1. Minimal strips and skillets.

**Definition 3.3** (Skillet). Let  $u: \mathbb{R} \rightarrow [0, +\infty]$  be a continuous compactly supported function such that  $u(x) = \infty$  if and only if  $|x| < 1$  and such that  $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 : y < u(x)\}$  has a uniformly smooth boundary, with  $u''(x) \geq 0$  along the boundary of  $\mathcal{A}$  (see Fig. 1.) Then the set

$$K = \text{closure}(\{(x, y) \in \mathbb{R}^2 : 0 < y < u(x)\})$$

is called a **skillet handle** with **edge**  $E = \{(x, y) \in K : y = 0\}$ . If  $K$  is a skillet handle with edge  $E$  and  $H$  is a vertical half-plane in  $\{z > 0\}$  such that  $K \cap H = K \cap \partial H = E$ , then we say that  $H \cup K$  is a **skillet**.

FIGURE 1. The boundary of a skillet and the minimal skillet  $M$ 

In order to establish the main results about bridges at infinity, we will need some results about existence, uniqueness and stability of area minimizing surfaces whose boundary at infinity is either a pair of straight lines or the boundary of a skillet. But first we need to introduce some ideas about stability.

Consider  $\Omega$  be a surface (or  $n$ -manifold) that is connected but not compact.

**Definition 3.4** ( $L^\infty$ -stability). Let  $J$  be a self-adjoint 2nd-order linear elliptic operator on a surface  $\Omega$ . Let's say  $\Omega$  is **strictly  $L^\infty$ -stable** (with respect to  $J$ ) if the first eigenvalue of any compact subdomain is strictly positive and if there are no nonzero bounded Jacobi fields (i.e. solutions of  $Ju = 0$ ) on  $\Omega$ .

The proof of the following lemma is standard (see, for example, Theorem 1 of [4]).

**Lemma 3.5.** *Let  $w$  be a positive solution of  $Jw = 0$  on  $\Omega$ . Then the first eigenvalue of  $w$  on every compact subdomain of  $\Omega$  is strictly positive.*

**Lemma 3.6.** *Let  $u$  and  $w$  be Jacobi fields on a connected minimal hypersurface  $M$ . Suppose that  $u/w$  has a positive local maximum  $\lambda$  at a point  $p$  where  $u$  and  $w$  are both positive. Then  $u = \lambda w$ .*

*Proof.* By hypothesis,  $u - \lambda w$  has a local maximum value 0. Thus by the strong maximum principle,  $u - \lambda w$  vanishes in a neighborhood of  $p$ . By the unique continuation property for solutions of second order elliptic equations,  $u - \lambda w \equiv 0$ .  $\square$

**Theorem 3.7.** *Suppose  $w$  is a positive solution of  $Jw = 0$  such that  $\lim_{p \rightarrow \partial\Omega} w(p) = \infty$ . Then  $\Omega$  is strictly  $L^\infty$ -stable.*

*Proof.* We have to show that each compact subdomain is stable and that there are no nonzero bounded Jacobi fields on  $\Omega$ . By Lemma 3.5, each compact subdomain is stable. Thus we need only show that there are no nonzero, bounded Jacobi fields.

Suppose  $u : \Omega \rightarrow \mathbb{R}$  is a nonzero, bounded Jacobi field on  $\Omega$ . We may suppose that  $u > 0$  at some points. Since  $u/w$  is positive at some points and tends to 0 on  $\partial\Omega$ , it has a local maximum  $\lambda > 0$  at some point  $\Omega$ . By Lemma 3.6,  $u \equiv \lambda w$ , which is impossible since  $u$  is bounded and  $w$  is unbounded.  $\square$

**Corollary 3.8.** *A totally geodesic plane in  $\mathbb{H}^3$  is strictly  $L^\infty$ -stable.*

*Proof.* Without loss of generality we can assume that the plane is a hemisphere centered at the origin in the upper halfspace model of  $\mathbb{H}^3$ . Consider the Jacobi field  $w$  that comes from dilations about 0.  $\square$

**Theorem 3.9.** *Let  $M$  be an area-minimizing surface in  $\mathbb{H}^3$  with  $\partial M \subset \partial_\infty \mathbb{H}^3$ . Let  $p$  be a regular point of  $\partial M$ , so that (in the upper halfspace model)  $M \cup \partial M$  is a regular manifold-with-boundary near  $p$ .*

*Let  $u$  be a bounded, nonnegative Jacobi field on  $M$ . Then  $\lim_{q \rightarrow p} u(q) = 0$ .*

*Proof.* Without loss of generality,  $p = 0$  in the upper half space model of  $H^3$ . Let  $p_n \in M$  be points such that  $p_n \rightarrow 0$  and such that

$$u(p_n) \rightarrow \limsup_{q \rightarrow 0} u(q).$$

Suppose the supremum limit is nonzero. Then we may assume it is 1. Now make a Euclidean translation and dilation of  $\mathbb{H}^3$  that moves  $M$  to  $M_n$  and that moves  $p_n$  to  $(0, 0, 1)$ . Let  $u_n$  be the Jacobi field on  $M_n$  corresponding to  $u$  on  $M$ . After passing to a subsequence, the  $M_n$  converge to a totally geodesic plane  $M^*$  and the  $u_n$  converge to a bounded Jacobi field  $u^*$  on  $M^*$  that attains its maximum value (1) at the point  $(0, 0, 1)$ . But that contradicts the strict  $L^\infty$ -stability of a totally geodesic plane.  $\square$

**Definition 3.10.** We say that a closed set  $K \subset \partial_\infty \mathbb{H}^3$  has *piecewise smooth boundary* provided

- (1)  $K$  is the closure of its interior, and
- (2) there is a finite set  $S$  of points such that  $(\partial K) \setminus S$  is the disjoint union of a finite set of smooth curves.

**Theorem 3.11.** *Let  $K \subset \partial_\infty \mathbb{H}^3$  be a closed region with piecewise smooth boundary. Then there is a least area surface  $M$  in  $\mathbb{H}^3$  such that  $\partial_\infty M = \partial K$ . Furthermore, if  $M$  is a least area surface in  $\mathbb{H}^3$  with  $\partial_\infty M = \partial K$ , then*

- (1)  *$\overline{M}$  is a smooth embedded manifold with boundary except at the finite set of points where  $\partial K$  is not a smooth embedded curve.*
- (2) *there is an open subset  $U$  of  $\mathbb{H}^3 \setminus M$  whose closure in  $\overline{\mathbb{H}^3}$  is  $K \cup M$ .*

*Proof.* Anderson [1, Theorem 3] proves existence of an area-minimizing surface  $M \subset H$  with the property that  $\partial M = \partial K$  as flat chains mod 2 with respect to the Euclidean metric on the ball. (He states the theorem for integral currents, but exactly the same proof works for chains mod 2.) In particular, this implies that  $\overline{M} \setminus M = \partial K$  as sets.

Now let  $M$  be any area-minimizing surface with  $\partial M \subset \partial K$ . Then  $M$  is smooth away from the boundary by the standard regularity theory. Hardt-Lin [5] prove that in a neighborhood  $U$  of each regular point of  $\partial K$ ,  $\overline{M} \cap U$  is a union of some finite number  $\kappa$  of  $C^1$  manifolds-with-boundary, the boundary being  $(\partial K) \cap U$ , and that those manifolds are disjoint except at the boundary. Their result is stated for integral currents, but their proof also works for chains mod 2 and in that case actually gives more:  $\kappa$  must then be 0 or 1 (because in Lemma 2.1 of their paper, if  $\delta$  is sufficiently small, then  $\kappa$  must be 0 or 1.) In our theorem, we are assuming that  $\partial M = \partial K$ , not just that  $\partial M \subset \partial K$ . Thus  $\kappa = 1$ . Tonegawa [8] improves the boundary regularity by showing that  $\overline{M}$  is  $C^\infty$  on the regular portions of  $\partial K$ .

Since  $M \cup K$  is a piecewise smooth, embedded (except possibly at finitely points) closed manifold in  $\mathbb{R}^3$ , there is an open subset  $W$  of  $\mathbb{R}^3$  such that  $\partial W = S$ . If  $W \subset \mathbb{B}$ , we let  $U = W$ . Otherwise, we let  $U = W^c$ .  $\square$

We say that  $U$  is the region enclosed by  $K$  and  $M$ , and we denote it by  $E(M, K)$ .

**Lemma 3.12.** *Let  $M$  be an area minimizing surface. Let  $M'$  be a compact region in the interior of  $M$  such that  $M'$  has piecewise smooth boundary. Then  $M'$  is the unique least area surface with its boundary.*

*Proof.* Standard.  $\square$

**Theorem 3.13.** *Let  $K_1$  and  $K_2$  be disjoint, closed regions in  $\partial_\infty \mathbb{H}^3$  with piecewise smooth boundaries. Let  $M_1$  and  $M_2$  be least area surfaces with boundaries  $\partial K_1$  and  $\partial K_2$ , and let  $U_i$  be the region enclosed by  $M_i \cup K_i$ . Then  $\overline{U_1}$  and  $\overline{U_2}$  are disjoint.*

*Proof.* Let  $Z = \overline{U_1} \cap \overline{U_2}$ . Note that  $Z$  is a compact subset of  $\mathbb{H}^3$ . Suppose it is nonempty. Then  $U_1 \cap U_2$  is nonempty by the maximum principle (applied to  $M_1$  and  $M_2$ .) By Lemma 3.12,  $U_1 \cap M_2$  is the unique least area surface with its boundary. Likewise,  $U_2 \cap M_1$  is the least area surface with its boundary. But  $U_1 \cap M_2$  and  $U_2 \cap M_1$  have the same boundary, a contradiction.  $\square$

**Corollary 3.14.** *Suppose for  $i = 1, 2$  that  $K_i$  is a closed region in  $\partial_\infty \mathbb{H}^3$  and that  $M_i$  is a least area surface in  $\mathbb{H}^3$  with  $\partial M_i = \partial K_i$ . Let  $U_i$  be the region enclosed by  $M_i \cup K_i$ . If  $K_1$  is contained in the interior of  $K_2$ , then  $U_1 \cup M_1$  is contained in  $U_2$ .*

(This corollary is not really a corollary – but it is proved in exactly the same way as the theorem. Actually, we use the corollary but not the theorem.)

**Theorem 3.15.** *Let  $K$  be a closed region in  $\partial_\infty \mathbb{H}^3$  with piecewise smooth boundary. Let  $\mathcal{F}$  be the collection of all least area surfaces in  $\mathbb{H}^3$  with boundary  $\partial K$ . Then  $\mathcal{F}$  contains surfaces  $M_{in}$  and  $M_{out}$  with the following property. If  $M \in \mathcal{F}$ , then*

$$E(M_{in}, K) \subset E(M, K) \subset E(M_{out}, K).$$

Recall the  $E(M, K)$  is the region enclosed by  $M$  and  $K$ .

*Proof.* Let  $K_1 \subset K_2 \subset \dots$  be a sequence of closed subsets of the interior of  $K$  such that each  $K_i$  has smooth boundary, such that  $\cup K_i$  is the interior of  $K$ , such that  $\partial K_i \rightarrow \partial K$ , and such that convergence  $\partial K_i$  to  $\partial K$  is smooth except at the points where  $\partial K$  is not smooth.

Let  $M_i$  be a least area surface with boundary  $\partial K_i$ , and let  $M_{in}$  be a subsequential limit of the  $M_i$ . Then  $M_{in} \in \mathcal{F}$ .

Furthermore, if  $M \in \mathcal{F}$ , then

$$E(M_i, K_i) \subset E(M, K)$$

for all  $i$  (by the lemma), and thus  $E(M_{in}, K) \subset E(M, K)$ .

The assertions about  $M_{out}$  are proved in a very analogous manner.  $\square$

**Remark 3.16.** Note that  $M_{in}$  is unique, as in  $M_{out}$ . Hence if  $g$  is an isometry of  $H$  such that  $g(K) = K$ , then  $g(M_{in}) = g(M_{in})$  and  $g(M_{out}) = M_{out}$ .

Of course  $M_{in} = M_{out}$  if and only there is only one least area surface with boundary  $K$ .

**Theorem 3.17.** *In the upper half space model of  $\mathbb{H}^3$ , let  $S$  be the strip*

$$\mathbb{R} \times [-1, 1] \times \{0\} = \{(x, y, z) : |y| \leq 1, z = 0\}$$

*together with the point at infinity.*

*Then there is a unique least area surface  $M$  with boundary  $\partial S$ , and  $M$  has the form*

$$\{(x, y, z) : z = u(y), |y| < 1\}$$

*where  $u : (-1, 1) \rightarrow \mathbb{R}$  is a smooth function such that*

$$\begin{aligned} u'' &< 0, \\ u(y) &\equiv u(-y), \\ \lim_{y \rightarrow \pm 1} u(y) &= 0. \end{aligned}$$

*Furthermore, the surface  $M$  is strictly  $L^\infty$ -stable.*

*Proof.* Note that each of the planes  $y = 1$  and  $y = -1$  is uniquely area minimizing, but their union is not area minimizing. It follows that any least area surface with boundary  $\partial K$  must be connected.

Let  $M_{in}$  and  $M_{out}$  be the innermost and outermost least area surfaces with boundary  $\partial K$ , as in Theorem 3.15.

Then (see Remark 3.16),  $M_{in}$  and  $M_{out}$  are both invariant under translations  $(x, y, z) \mapsto (x + c, y, z)$ . It follows that

$$C_{in} := \{(y, z) : (0, y, z) \in M_{in}\},$$

and

$$C_{out} := \{(y, z) : (0, y, z) \in M_{out}\},$$

are smooth, connected curves. Since  $M_{in}$  is connected and joins  $(-1, 0)$  to  $(1, 0)$ ,  $M_{in}$  has compact closure, and similarly for  $M_{out}$ .

Now if  $M_{in} \neq M_{out}$ , there is some  $\lambda > 0$  such that  $\lambda C_{in}$  intersects  $\lambda C_{out}$ . Thus there is a largest  $\lambda$  (since  $C_{in}$  and  $C_{out}$  have the same endpoints and have compact closures.) But then  $\lambda M_{in}$  and  $M_{out}$  violate the maximum principle.

Thus there is a unique least area surface  $M = M_{in} = M_{out}$  with boundary  $\partial K$ .

Now where the tangent to the curve  $C = C_{in} = C_{out}$  is not vertical, it is locally the graph of a function  $z = u(y)$  that satisfies a 2nd order ODE, namely  $u(y) \cdot u''(y) + 2(1 + (u'(y))^2) = 0$ , from which we see that  $u'' < 0$  and thus that  $C$  has the form

$$C = \{(0, y, u(y)) : |y| < 1\}, \quad \lim_{y \rightarrow \pm 1} u(y) = 0.$$

By Remark 3.16,  $M$  is invariant under  $(x, y, z) \mapsto (x, -y, z)$  and hence the function  $u$  is even.

So, summarizing all the information that we have, we are able to deduce that  $y \cdot u(y) < 0$ ,  $-1 < y < 1$ . Furthermore, we know that

$$\lim_{y \rightarrow -1} u'(y) = +\infty, \quad \text{and} \quad \lim_{y \rightarrow +1} u'(y) = -\infty.$$

Let  $w^*$  be the Jacobi field on  $M$  associated to dilations  $(x, y, z) \mapsto \lambda(x, y, z)$ . Then  $w^*$  is strictly positive everywhere, so compact domains in  $M$  are strictly stable. A straightforward computation gives

$$w^* = \frac{-yu' + u}{u\sqrt{1 + (u')^2}},$$

so

$$(3.1) \quad w^* \rightarrow \infty \text{ uniformly as } y \rightarrow \pm 1.$$

Now suppose that  $M$  is not  $L^\infty$  strictly stable, i.e., that  $M$  has a bounded, nonzero Jacobi field  $v$ . We may assume that  $v$  is strictly positive at some points. Let  $\Lambda$  be the supremum of  $v/w^*$ , and let  $p_n := (x_n, y_n, z_n) \in M$  be a sequence of points such that

$$v(p_n)/w^*(p_n) \rightarrow \Lambda.$$

By (3.1), the  $|y_n|$  is bounded away from 1. Thus by passing to a subsequence, we can assume that the points  $(0, y_n, z_n)$  converge to a point  $p \in M$  and that the jacobi fields  $(x, y, z) \mapsto v(x - x_n, y, z)$  converge smoothly to a limit jacobi field  $\hat{v}$ . Note that  $\hat{v}/w^*$  attains its maximum value  $\Lambda$  at  $p$ . Thus the jacobi field  $\hat{v} - \Lambda \cdot w^*$  attains its maximum value, namely 0, at  $p$ . By the maximum principle,  $\hat{v} - \Lambda \cdot w^*$  must be identically 0. But that is impossible since  $\hat{v}$  is bounded and  $\Lambda w^*$  is unbounded.  $\square$

**Theorem 3.18.** *Let  $H \cup S$  be a skillet in  $\mathbb{H}^3$  and let  $W = \partial_\infty(H \cup S)$ . Then there exists a properly embedded uniquely area minimizing surface  $M$  satisfying  $\partial_\infty M = W$ . Furthermore, the surface  $M$  is diffeomorphic to  $H \cup S$ , and there are no points in  $M$  with vertical normal vector.*

*Proof.* Without loss of generality we can assume that the skillet handle is contained in  $\{z = 0, y > 0\}$  and that edge of the handle  $E \subset \{y = 0\}$ . We consider a sequence of regular, simple closed curves  $\{\Gamma_n\}_{n \in \mathbb{N}}$  obtained by a suitable regularization of

$$\{(x, y, 0) \in \partial_\infty(H \cup S) : y \leq n\} \cup \{(x, n, 0) : |x| \leq 1\}.$$

See Figure 2.



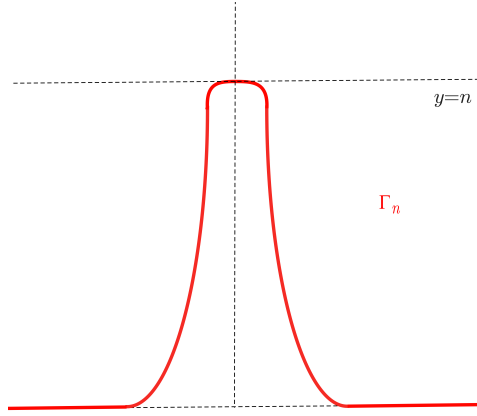


FIGURE 2. The curve  $\Gamma_n$  consists of a regularization of  $\{(x, y, 0) \in \partial_\infty(H \cup S) : y \leq n\} \cup \{(x, n, 0) : |x| \leq 1\}$ .

Using Lin's results [6] we have that there exists a unique area minimizing (mod 2) surface  $M_n$  satisfying  $\partial_\infty M_n = \Gamma_n$ . Moreover, Hardt and Lin [5] proved that, since our surfaces bound a star-shape domain  $\Omega$  with respect to the point  $p_\epsilon := (0, -\epsilon, 0)$ ,  $\epsilon > 0$ , then  $M_n$  is a *radial graph* of a function defined on an open hemisphere centered at  $p_\epsilon$ .

As  $\{M_n\}_{n \in \mathbb{N}}$  is a sequence of area minimizing surfaces mod 2 then we have area estimates. So, up to a subsequence,  $M_n$  converges, smoothly on compact sets of  $\mathbb{H}^3$ , to an area minimizing surface  $M$ . From our method of construction, it is clear that  $\partial_\infty M = W$ , and that  $M$  is also a radial graph, with respect to the point  $p_\epsilon$ , for any  $\epsilon > 0$ . Taking limit as  $\epsilon \rightarrow \infty$ , we deduce that  $M$  is a (horizontal) graph over the half-plane  $\{y = 0, z > 0\}$ . This implies that there are no points whose normal lies in the plane  $\{y = 0\}$ . Indeed, suppose there exists  $(x_1, y_1, z_1) \in M$  such that  $\nu(x_1, y_1, z_1) \in \{x = 0\}$ . Then we define  $M_+ \stackrel{\text{def}}{=} M \cap \{x \leq x_1\}$ ,  $M_- \stackrel{\text{def}}{=} M \cap \{x \geq x_1\}$  and  $M_+^*$  the reflection of  $M_+$  with respect to the plane  $\{x = x_1\}$ . Using the boundary maximum principle at  $(x_1, y, z_1)$  we would deduce that  $M_+^* = M_-$ , which is absurd because  $W = \partial_\infty M$  does not have such a symmetry. In particular, there are no points in  $M$  with vertical normal vector.

**Claim 3.19.** *As  $x^2 + z^2 \rightarrow \infty$ , the skillet is asymptotic to the halfplane  $y = 0, z \geq 0$ .*

Take a sequence of dilations  $(x, y, z) \mapsto \frac{1}{n}(x, y, z)$ . The images  $M_n$  of the skillet under those dilations converge to an area minimizing surface  $M'$  whose ideal boundary is the  $x$ -axis plus (perhaps) some or all of the positive  $y$ -axis.

But in fact the positive  $y$ -axis is not part of the ideal boundary, by the even-odd argument. Take a point  $p_0 = (0, y_0, 0)$  with  $y_0 > 0$  and consider the plane  $\Pi_0 = \{y = y_0\}$ . Then, the (Euclidean) circle  $C_R = \{x^2 + z^2 = R, y = y_0\}$  must intersect  $M'$  in an even number of points, for any  $R$ .

We claim that  $C_R$  does not intersect  $M'$ , when  $R$  is sufficiently small. If not, it is possible to find a sequence of radii  $\{R_j\} \searrow 0$  such that  $C_{R_j} \cap M' \neq \emptyset$ , for all  $j \in \mathbb{N}$ . Let  $M'(j)$  be the image under the isometry  $x \mapsto (1/R_j)(x - p_0) + p_0$  of  $M'$ . Thus, up to a subsequence,  $\{M'(j)\}_{j \in \mathbb{N}}$  converges to an area minimizing surface  $M_0$

whose ideal boundary is the  $y$ -axis. Hence,  $M_0$  is the half-plane  $\{x = 0, z > 0\}$ . From the above construction we have that the circle  $C_1$  intersects  $M_0$  in an even number of points, which is absurd. This means that  $C_R$  does not intersect  $T$ , when  $R$  is sufficiently small, and so  $p_0 \notin \partial_\infty M'$ .

Thus the ideal boundary of  $M'$  is the  $x$ -axis, so  $M'$  is the half-plane  $\{y = 0, z > 0\}$ .

Given  $\rho > 0$  we define

$$d(\rho) := \sup\{\text{dist}_{\mathbb{H}^3}(p, M') \mid p \in M \cap \{x^2 + z^2 \geq \rho\}\}.$$

Then, we have that  $\lim_{\rho \rightarrow +\infty} d(\rho) = 0$ . If not, there exists  $\epsilon > 0$  and a sequence of points  $\{p_n = (x_n, y_n, z_n)\}$  in  $M$ , such that:

- $\rho_n = x_n^2 + z_n^2$  diverges.
- $\text{dist}_{\mathbb{H}^3}(p_n, M') \geq \epsilon$ .

Reasoning as above, the sequence  $\frac{1}{\rho_n} \cdot M$  converges to  $M'$  smoothly on compact sets of  $\mathbb{H}^3$ . But the sequence of points  $\{p_n/\rho_n\}$  converges to a point  $p' = (x', y', z') \in M'$  with  $x'^2 + z'^2 = 1$  and  $\text{dist}_{\mathbb{H}^3}(p', M') \geq \epsilon$  which is absurd. This contradiction completes the proof of Claim 3.19.

**Claim 3.20.**  *$M$  is asymptotic to the surface given by Theorem 3.17, as  $y \rightarrow \infty$ ,  $x^2 + z^2$  bounded.*

Indeed, if we translate the surface  $M$  by  $(0, -a, 0)$ ,  $a > 0$ , then the limit surface as  $a \rightarrow +\infty$  is an area minimizing surface whose ideal boundary is  $\mathbb{R} \times \{-1, 1\}$ . Using Theorem 3.17 and reasoning as in the proof of Claim 3.19 we conclude this claim.

In order to prove the uniqueness, assume there is another surface  $T$  satisfying the hypothesis of the theorem. Reasoning as before, we can deduce that  $T$  also satisfies Claim 3.20. Fix  $R > 0$  sufficiently large so that  $p_\epsilon \in B(0, R)$  and  $M \setminus B(0, R)$  consists of two connected components (where  $B(0, R)$  means the Euclidean ball in  $\mathbb{R}^3$ ). Let  $\Omega$  be the connected component that does not touch the skillet handle. Denote by  $T_\lambda$  the result of dilating  $T$  by  $\lambda$  from  $p_\epsilon$ . Taking into account the asymptotic behavior of  $M$  and  $T$ , it is clear that  $M \cap T_\lambda$  is contained  $\Omega \setminus \partial\Omega$ . Furthermore, we have  $M \cap T_\lambda$  cannot approach  $\partial_\infty M = \partial_\infty T = W$ . By the maximum principle for minimal surfaces,  $M \cap T_\lambda$  cannot have compact connected components. Take an unbounded curve  $\Gamma \subset M \cap T_\lambda$  spanning a (unbounded) minimal surface  $\Sigma_\lambda \subset T_\lambda$ .

Consider  $\Omega^*$ ,  $\Sigma_\lambda^*$ , and  $\Gamma^*$  the result of inverting  $\Omega$ ,  $\Sigma_\lambda$ , and  $\Gamma$  (respectively) with respect the sphere  $\mathbb{S}_0^2(R) = \{p \in \mathbb{R}^3 : \|p\| = R\}$ . Thus, we have:

- $\Omega^*$  is a minimal disk with  $\partial\Omega^* = \partial\Omega \subset (\mathbb{S}_0^2(R) \cap \{z > 0\})$  and  $\partial_\infty\Omega^* = \{(0, y, 0) : |y| \leq R\}$ .
- $\Gamma^*$  is a curve in  $\Omega^*$  such that  $\Gamma \cap \partial_\infty\mathbb{H}^3 = \{0\}$ . If  $\lambda$  is sufficiently large, then the orthogonal projection of  $\Gamma^*$  over the half plane  $\{x = 0, z > 0\}$  does not intersect the orthogonal projection of  $\partial\Omega^*$ .
- $\Sigma_\lambda^*$  is a minimal surface with boundary  $\Gamma^*$ .

Define  $\Omega_t^*$  the surface obtained by applying the translation  $(x, y, z) \mapsto (x + t, y, z)$  to  $\Omega^*$ . Consider  $t_0 = \sup\{t > 0 : \Omega_t^* \cap \Sigma_\lambda^* \neq \emptyset\}$ , then it is clear that  $\Omega_{t_0}^*$  and  $\Sigma_\lambda^*$

have an interior point of contact, which is absurd. This contradiction proves that  $M$  and  $T$  are equal.  $\square$

**Definition 3.21.** The minimal surface  $M$  in Theorem 3.18 is called a *minimal skillet*.

**Theorem 3.22.** A minimal skillet  $M$  in  $\mathbb{H}^3$  is strictly  $L^\infty$ -stable.

*Proof.* Consider the upper halfspace model  $\mathbb{H}^3 \cong \{(x, y, z) : z > 0\}$ . We may assume that our skillet  $M$  lies in the region  $\{(x, y, z) : z > 0 \text{ and } y > 1\}$ . As  $x^2 + z^2 \rightarrow \infty$ , the skillet is asymptotic (Claim 3.19) to the halfplane  $y = 1, z \geq 0$ . As  $y \rightarrow \infty$  with  $x^2 + z^2$  bounded, the skillet is asymptotic to a surface as in Theorem 3.17 (see Claim 3.20.)

Let  $w$  be the Jacobi field on  $M$  corresponding to dilations about 0. In other words, for  $p \in M$ ,  $w(p)$  is the (hyperbolic) length of  $p^\perp$ . Then  $w > 0$  everywhere, so compact subsets of  $M$  are strictly stable. Thus it suffices to show that  $M$  has no nonzero, bounded Jacobi fields.

Suppose to the contrary that  $v$  is a nonzero, bounded Jacobi field.

**Claim 3.23.**  $zw(x, y, z)$  is bounded away from 0.

*Proof of Claim 3.23.* Note that  $w(x, y, z)$  is the hyperbolic length of the vector  $(x, y, z)^\perp$  at the point  $p$ , so  $zw(x, y, z)$  is the Euclidean length  $|(x, y, z)^\perp|$  of  $(x, y, z)^\perp$ .

As  $x^2 + z^2 \rightarrow \infty$  in  $M$ ,  $\text{Tan}_{(x, y, z)} M$  converges to the plane  $y = 0$ , so  $(x, y, z)^\perp \sim (0, y, 0)$ . Also,  $y \geq 1$  on  $M$ , so

$$\liminf_{x^2+z^2 \rightarrow \infty} zw(x, y, z) \geq 1.$$

On sets where  $x^2 + z^2$  and  $y$  are both bounded, the Euclidean length of  $(x, y, z)^\perp$  is bounded away from 0 because  $M$  is a radial graph.

Thus it remains to show that the Euclidean length  $|(x, y, z)^\perp|$  is bounded as  $y \rightarrow \infty$  with  $x$  and  $z$  bounded. But that follows from the fact that  $M$  is asymptotic as  $y \rightarrow \infty$  to the surface described in Theorem 3.17. This completes the proof of the Claim 3.23.  $\square$

**Claim 3.24.** If  $p_n = (x_n, y_n, z_n)$  is a divergent sequence in  $M$ , then  $v(p_n) \rightarrow 0$ .

*Proof of Claim 3.24.* By passing to a subsequence, we can assume that one of the following holds:

- (1)  $(x_n)^2 + (z_n)^2 \rightarrow \infty$ .
- (2)  $(x_n)^2 + (z_n)^2$  is bounded and  $z_n \rightarrow 0$ .
- (3)  $(x_n)^2 + (z_n)^2$  is bounded and  $z_n$  is bounded away from 0.

Translate  $M$  by  $(-x_n, -y_n, 0)$  and then dilate by  $1/z_n$  to get a surface  $M_n$ . Let  $v_n$  be the Jacobi field on  $M_n$  corresponding to  $v$  on  $M$ . By passing to a subsequence, we can assume that the  $M_n$  converges smoothly to a limit surface  $\hat{M}$ , and that the  $v_n$  converge to a bounded Jacobi field  $\hat{v}$  on  $\hat{M}$ . In case (1),  $\hat{M}$  is the totally geodesic halfplane  $\{y = 0\}$  (by Claim 3.19). In case (2),  $\hat{M}$  is also a totally geodesic halfplane, since  $\partial_\infty \hat{M}$  is a line. In case (3),  $\hat{M}$  is (up to a translation and dilation), the surface described in Theorem 3.17. In all three cases,  $\hat{M}$  is strictly stable. Thus  $\hat{v} = 0$ . Since  $\hat{v}(0, 0, 1) = \lim v_n(0, 0, 1) = \lim v(p_n)$ , this completes the proof of Claim 3.24.  $\square$

**Claim 3.25.** *There exists a Jacobi field  $f$  on  $M$  and an  $R > 0$  such that*

$$\inf_{M \cap \{x^2 + z^2 > R\}} f > 0.$$

*Proof of Claim 3.25.* Let  $S_R$  be the surface  $S$  obtained from  $M \cap \{x^2 + z^2 > R\}$  by inversion in the sphere  $x^2 + y^2 + z^2 = 1$ . Note that if  $R$  is sufficiently large, then  $S \cup \partial_\infty S$  is a smooth surface on which  $y$  is a smooth function of  $x$  and  $z$ . Indeed, by choosing  $R > 0$  sufficiently large, we can guarantee that the Euclidean unit normal to  $S$  is everywhere arbitrarily close to  $(0, 1, 0)$ . Consequently, the Jacobi field corresponding to translations in the  $y$ -direction is bounded away from 0 on  $S$ . Now let  $f$  be the corresponding one on  $M$ . This completes the proof of Claim 3.25.  $\square$

Now let  $\lambda = \sup(v/w)$ . Since we are assuming that  $v > 0$  at some points,  $\lambda > 0$ . By Lemma 3.6, the supremum is not attained at any point of  $M$ . (Note that  $v$  cannot be a multiple of  $w$  since  $v$  is bounded and  $w$  is unbounded.) Thus if  $p_n = (x_n, y_n, z_n)$  is a sequence of points in  $M$  with  $v(p_n)/w(p_n) \rightarrow \lambda$ , then  $p_n$  diverges in  $M$ . By claim 1,  $v(p_n) \rightarrow 0$ . Since  $\lambda > 0$ , this implies that  $w(p_n) \rightarrow 0$ , and therefore by Claim 3.23 that  $z_n \rightarrow \infty$ .

It follows that by choosing  $\mu < \lambda$  sufficiently close to  $\lambda$ , we can guarantee that

$$(v - \mu w)^+$$

is supported in  $M \cap \{z > R\}$ , where  $R$  is as in Claim 3.25. It follows (using Claims 3.24 and 3.25) that

$$(v - \mu w)^+ / f$$

attains a positive maximum value  $k$  at some point  $p$ . Consequently,

$$(v - \mu w) / f$$

has a positive local maximum  $k$  at  $p$ , so

$$v - \mu w - kf \equiv 0$$

by Lemma 3.6. But that is impossible since  $(v - \mu w)$  is negative at some points of  $M \cap \{z > R\}$  whereas  $f > 0$  everywhere on that set. The contradiction proves that there is no such  $v$ , and therefore that  $M$  is strictly  $L^\infty$  stable.  $\square$

**3.2. The proof of the bridge principle.** Let  $M \subset \mathbb{H}^3$  be a smooth properly embedded surface that admits a smooth extension to  $\overline{\mathbb{H}^3}$ . Let  $\Gamma \subset \partial_\infty \mathbb{H}^3$  be a smooth embedded arc such that  $\overline{M} \cap \Gamma = \partial\Gamma$  and  $\Gamma$  meets  $\partial_\infty M$  orthogonally at either of its ends points. A **bridge on  $M$  along  $\Gamma$**  is the image  $P$  of a homeomorphism

$$\phi : [0, 1] \times [-1, 1] \longrightarrow \partial_\infty \mathbb{H}^3$$

such that  $\phi(\cdot, 0)$  parametrizes  $\Gamma$  and  $\phi(t, s) \in \overline{M}$  if, and only if,  $t = 0$  or  $t = 1$ .

By the (Euclidean) **width** of  $P$  we mean

$$w(P) = \sup_{x \in P} \text{dist}_{\mathbb{R}^3}(x, \partial P).$$

For the following proposition, we shall consider the *half-space model* of  $\mathbb{H}^3$ . In this model, the homotheties centered at points  $p \in \{z = 0\}$  induce isometries of the hyperbolic space.

**Proposition 3.26.** *Let  $M$  and  $\Gamma$  be as in the previous proposition. Then, there exists a sequence of bridges  $\{P_n\}_{n \in \mathbb{N}}$  on  $M$  along  $\Gamma$  satisfying:*

(a) *The widths  $w_i := w(P_i)$  tends to 0, as  $i \rightarrow \infty$ ;*

(b) The symmetric difference  $(\partial P_i) \Delta \partial M$  is smooth and if  $x_i \in P_i$ , then for any sequence of  $i$ 's tending to  $\infty$  has a subsequence  $\Lambda$  so that

$$\lim_{i \in \Lambda} (w_i^{-1})_{\#} ((\partial P_i) \Delta \partial M - x_i)$$

converges smoothly on compact sets of  $\overline{\mathbb{H}^3}$  to either:

- (1) two parallel straight lines,
- (2) the boundary of a **skillet**.

Recall that  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .

**Theorem 3.27.** *Let  $S$  be a properly embedded, uniquely area minimizing surface in  $\mathbb{H}^3$  that extends  $C^\infty$  to an embedding into  $\overline{\mathbb{H}^3}$ , that we call  $\overline{S}$ . Assume that  $\overline{S}$  has finite topology. Let  $\Gamma$  be a smooth arc in  $\partial_\infty \mathbb{H}^3$  with meets  $\partial_\infty S$  orthogonally and satisfying  $\Gamma \cap \partial_\infty S = \partial \Gamma$ .*

*Consider a sequence of bridges  $P_n$  on  $\partial_\infty \mathbb{H}^3$  that shrink nicely to  $\Gamma$ . If  $S$  is **strictly  $L^\infty$ -stable**, then for all large enough  $n$ , there exists a strictly  $L^\infty$ -stable, uniquely area minimizing surface  $S_n$  that is properly embedded in  $\mathbb{H}^3$  and that satisfies:*

- 1)  $\partial_\infty S_n = \partial_\infty S \Delta \partial P_n$  (in particular,  $S_n$  extends  $C^\infty$  to  $\overline{\mathbb{H}^3}$ );
- 2) The sequence  $S_n$  smoothly converges to  $S$  on compact subsets of  $\mathbb{H}^3$ ;
- 3) The surface  $\overline{S}_n$  is homeomorphic to  $\overline{S} \cup P_n$

*Proof.* Proposition 3.26 gives us a sequence of bridges  $\{P_i\}_{i \in \mathbb{N}}$  on  $S$  along  $\Gamma$  satisfying (1) and (2).

Fix  $i \in \mathbb{N}$  and consider the curve  $W_i := (\partial S) \Delta (\partial P_i)$ . Using some results by M. Anderson [1, 2] we know that there exists an embedded, absolutely area minimizing mod 2,  $S_i \subset \mathbb{H}^3$ , which is asymptotic to  $W_i$  at infinity. Furthermore, results by Tonegawa [8] guarantee that the surface  $S_i$  has the same regularity at infinity as  $\Gamma$ . In other words,  $S_i$  admits a smooth extension  $\overline{S}_i$  to  $\overline{\mathbb{H}^3}$ .

As  $\{S_i\}_{i \in \mathbb{N}}$  is a sequence of area minimizing mod 2 surfaces, then we have area estimates that allow us to obtain a subsequence—that we still label  $\{S_i\}_{i \in \mathbb{N}}$ —which converges on compact sets of  $\mathbb{H}^3$  to an area minimizing surface  $T$ . By its own construction, we know that

$$\partial_\infty S \subseteq \partial_\infty T \subseteq (\partial_\infty S \cup \Gamma).$$

Moreover, a standard argument gives us that  $\Gamma \setminus \partial \Gamma$  is not contained in  $\partial_\infty T$ .

Take a point  $p \in \Gamma \setminus \partial \Gamma$  and consider  $\Pi$  the normal plane to  $\Gamma$  at  $p$ . Then, the (Euclidean) circle  $C_R = \{q \in \Pi : \|q - p\| = R\}$  must intersect  $T$  in an even number of points, for any generic  $R$ . Assume  $p \in \partial_\infty T$ . Then it is possible to find a sequence of radii  $\{R_j\} \searrow 0$  such that  $C_{R_j} \cap T \neq \emptyset$ , for all  $j \in \mathbb{N}$ . Let  $T(j)$  and  $\Gamma(j)$  be the image under the isometry  $x \mapsto (1/R_j)(x - p) + p$  of  $T$  and  $\Gamma$ , respectively. Thus, we have a sequence  $\{T(j)\}_{j \in \mathbb{N}}$  of area minimizing mod 2 surfaces that (up to a subsequence) converges to an area minimizing surface  $T_0$  whose ideal boundary is the limit of  $\Gamma(j)$ : a straight line at  $\partial_\infty \mathbb{H}^3$  passing through  $p$ . Hence,  $T_0$  is a vertical half plane orthogonal to  $\Pi$ . From the above construction we have that the circle  $C_1$  intersects  $T_0$  in an even number of points, which is absurd. This means that  $C_R$  does not intersect  $T$ , when  $R$  is sufficiently small, and so  $p \notin \partial_\infty T$ .

So,  $T$  is an area minimizing surface satisfying  $\partial_\infty T = \partial_\infty S$ . Taking into account that  $S$  is *uniquely area minimizing*, then we have  $T = S$ .

Now, we shall prove that  $\overline{S_n}$  and  $\overline{S} \cup P_n$  are homeomorphic. The surface  $\overline{S_n}$  separates  $\mathbb{H}^3$  into two connected components, one of them contains the curve  $\Gamma$  which we denote by  $\mathcal{Q}_n$ .

For  $a > 0$ , we define  $\mathcal{R}_a := \{(x, y, z) \in \mathbb{H}^3 : 0 \leq z \leq a\}$ .

**Claim 3.28.** *There exists  $a > 0$  such that  $\overline{S_n} \cap \mathcal{R}_a$  does not contain critical points of  $z$  where  $u = (0, 0, 1)$  is the normal vector pointing toward  $\mathcal{Q}_n$ .*

We proceed by contradiction. Suppose this were not the case. Thus, after passing to a subsequence, we can assume that there exists a critical point  $p_n = (x_n, y_n, z_n) \in S_n$  with  $u$  points toward  $\mathcal{Q}_n$  at  $p_n$  and with  $z_n \rightarrow 0$ . Up to a subsequence, we can suppose that  $\{p_n\}$  converges to some point  $p_0 = (x_0, y_0, 0) \in \partial_\infty \mathbb{H}^3$ .

Then, we apply the isometry  $(x, y, z) \mapsto 1/z_n((x, y, z) - (x_0, y_0, 0))$  to  $S_n$  and  $p_n$  to obtain a new surface  $S'_n$  and a point  $p'_n \in S'_n$ . Label  $\Gamma(n)$  the image of  $\Gamma$  under the above homothety. Up to a subsequence, the sequences  $S'_n$ ,  $p'_n$  and  $\partial_\infty S'_n$  converge to limits  $S'$ ,  $p' \in S'$  and  $C'$ , respectively. Note that  $p'$  is a critical point of the function  $z$ . Furthermore, by the nice shrinking, we have either

- (i)  $C'$  is a line.
- (ii)  $C'$  is the union of a line and a perpendicular half-line, forming a T-shape.
- (iii)  $C'$  is the boundary of a skillet.
- (iv)  $C'$  is the union of two parallel lines.

In cases (i) and (ii)  $S'$  would be a vertical half-plane and in case (iii)  $S'$  would be a skillet. But half planes and skillets do not contain critical points of  $z$ , then these cases are not possible. So,  $S'$  is a minimal strip. The curves  $\Gamma(n)$  converge to the straight line  $\Gamma'$  in  $\partial_\infty \mathbb{H}^3$  which is parallel to the two straight lines in  $\partial_\infty S'$  and is equidistant to both lines. So,  $p'$  is a critical point of  $z$  where  $u$  points toward the region bounded by  $S'$  that contains  $\Gamma'$ , which is absurd. This contradiction proves the claim.

**Claim 3.29.** *The surfaces  $\overline{S_n}$  and  $\overline{S} \cup P_n$  are homeomorphic.*

Suppose  $\overline{S_n}$  is not homemorphic to  $\overline{S} \cup P_n$ . As they have the same boundary, then it means that  $\overline{S_n}$  and  $\overline{S} \cup P_n$  have different genus. Consider the positive constant  $a$  given by Claim 3.28. The smooth convergence on compact sets implies  $S_n \cap (\mathbb{H}^3 \setminus \mathcal{R}_a)$  is homemorphic to  $S \cap (\mathbb{H}^3 \setminus \mathcal{R}_a)$ , so our assumption gives that  $S_n \cap \mathcal{R}_a$  has non trivial genus.

Up to a slight modification of the point of infinity in the upper half-space model of  $\mathbb{H}^3$ , we can assume that the function  $z$  is a **Morse function** for the surface  $S_n$ . This implies the existence of a critical point of the height function  $z$  in  $\overline{S_n} \cap \mathcal{R}_a$  such that the vector  $u = (0, 0, 1)$  points in the direction of the region  $\mathcal{Q}_n$ , which is contrary to Claim 3.28. This contradiction completes the proof of this claim.

**Claim 3.30.** *If  $n$  is large enough, then the surfaces  $S_n$  are unique, i.e., if  $T_n$  is any area minimizing surface in  $\mathbb{H}^3$  with  $\partial_\infty T_n = \partial_\infty S_n$ , then  $S_n = T_n$  (for sufficiently large  $n$ ). Furthermore,  $S_n$  is strictly  $L^\infty$ -stable.*

Suppose the uniqueness is false. Then, up to a subsequence, we may assume that  $S_n$  and  $T_n$  are different  $\forall n \in \mathbb{N}$ . As  $S_n$  and  $T_n$  are asymptotic at  $\partial_\infty \mathbb{H}^3$ , then we can find  $q_n = (x_n, y_n, z_n) \in S_n$  that maximizes the (hyperbolic) distance to  $T_n$ .

On the other hand, all the claims we have already proved for  $S_n$  are also true for the surfaces  $T_n$ . In particular, using similar arguments, we can deduce that the

surfaces  $T_n$  smoothly converge to  $S$  on compact sets of  $\mathbb{H}^3$ . Hence, we have that:

$$(3.2) \quad \lim_{n \rightarrow \infty} \text{dist}(q_n, T_n) = 0.$$

Notice first that the third coordinate of  $q_n$  must converge to 0, as  $n \rightarrow \infty$ . If not, after passing to a subsequence, we may assume that  $\{q_n\}_{n \in \mathbb{N}}$  converges to a point  $q_0 \in S$ . From (3.2) it follows that there exists a normal vector field  $V_n$  defined on  $S_n$  such that a portion of  $T_n$  is a graph of  $V_n$ . After taking a subsequence again, the normal vector fields  $U_n = V_n / \|V_n(q_n)\|$  converge smoothly on compact sets of  $\mathbb{H}^3$  to a bounded Jacobi field  $U$  on  $S$ . Notice that  $\|U(q_0)\| = 1$ , but this fact contradicts the strict stability of  $S$  and so it proves that

$$(3.3) \quad \lim_{n \rightarrow \infty} z_n = 0.$$

(3.3) implies that, after passing to a subsequence, we can assume that  $q_n$  converges to a point  $q_\infty \in \Gamma$ . Now we translate  $S_n$  and  $T_n$  by  $-q_\infty$  and dilate by  $1/z_n$  in order to obtain new surfaces  $S'_n$  and  $T'_n$ . Up to a subsequence, we assume that they converge to surfaces  $S'$  and  $T'$ , respectively. From the choice of  $q_\infty$ , we have that  $S'$  and  $T'$  are either a vertical half-space, or skillet-like minimal surfaces or infinite minimal strips. As they have the same ideal boundary, then Theorems 3.17 and 3.18 give us that  $S' = T'$ . Since the convergence is smooth, then  $T'_n$  can be written as a graph over  $S'_n$  of a normal vector field  $V_n$ . By choice of  $q_n$ , the field  $V_n / \|V_n(q_n)\|$  converges to a non-zero bounded Jacobi field on  $S'$ . But this is contrary to the results contained in Corollary 3.8, Theorem 3.17 and Theorem 3.18. This proves the uniqueness.

In order to obtain the strict stability, we proceed again by contradiction. We can assume, up to a subsequence, that strict stability fails for all  $n \in \mathbb{N}$ . Then we have the existence of a non-zero bounded Jacobi field  $V_n$  on  $S_n$ . By Theorem 3.9, we know that  $V_n$  tends to zero at infinity. So,  $V_n$  attains its maximum at a point  $q_n$ . At this point we can argue like in the proof that  $S_n = T_n$ .  $\square$

#### 4. PROPERLY EMBEDDED AREA MINIMIZING SURFACES IN $\mathbb{H}^3$

In this section, we are going to prove the main existence results for properly embedded area minimizing surfaces with arbitrary (orientable) topology. The techniques we use are inspired in those developed by Ferrer, Meeks and the first author for the study of the Calabi-Yau problem in  $\mathbb{R}^3$  (see [3]).

**Theorem 4.1.** *Let  $S$  be an open oriented surface. Then, there exists a complete, proper, area minimizing embedding  $\psi : S \rightarrow \mathbb{H}^3$ . Moreover, the embedding  $\psi$  can be constructed in such a way that the limit sets of different ends of  $S$  are disjoint.*

*Proof.* Along this proof we are going to use the model of the Poincaré ball. Let  $S = \{S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots\}$  be a simple exhaustion of  $S$ . Our purpose is to construct a sequence of properly embedded minimal surfaces  $\{\Sigma_n\}_{n \in \mathbb{N}}$  and two sequences of positive real numbers  $\{\varepsilon_n\}_{n \in \mathbb{N}}$  and  $\{r_n\}_{n \in \mathbb{N}}$  satisfying:

- (1)  $\{\varepsilon_n\} \searrow 0$  and  $\{r_n\} \nearrow +\infty$ ;
- (2)  $\sum_{n=1}^{\infty} \varepsilon_n < 1$  and  $\sum_{n=i+1}^{\infty} \varepsilon_n < \delta_i$ , for all  $i \in \mathbb{N}$ ;

Moreover, for each  $n \in \mathbb{N}$ , the minimal surface  $\Sigma_n$  verifies:

- (I<sub>n</sub>)  $\Sigma_n$  is *strictly stable* and uniquely area minimizing;
- (II<sub>n</sub>)  $\Sigma_n$  admits a  $C^\infty$  extension  $\bar{\Sigma}_n$  to  $\bar{\mathbb{H}}^3$  so that  $\bar{\Sigma}_n$  is diffeomorphic to  $S_n$ ;
- (III<sub>n</sub>)  $\Sigma_n \cap \overline{B(0, r_j)}$  is diffeomorphic to  $S_j$ , for  $j = 1, \dots, n$ , where  $B(0, r)$  represents the hyperbolic ball centred at 0 of radius  $r$ ;
- (IV<sub>n</sub>)  $\Sigma_n \cap B(0, r_i)$  is a normal graph over its projection  $\Sigma_{i,n} \subset \Sigma_i$ , for  $i < n$ . Furthermore, if we write  $\Sigma_n \cap B(0, r_i) = \{\exp_p(f_{i,n}(p) \cdot \nu_i(p)) \mid p \in \Sigma_{i,n}\}$ , where  $\nu_i$  is the Gauss map of  $\Sigma_i$ , then:
  - $|\nabla f_{i,n}| \leq \sum_{k=i+1}^n \varepsilon_k$  and
  - $|f_{i,n}| \leq \sum_{k=i+1}^n \varepsilon_k$ , for  $i = 1, \dots, n-1$ .

First, we fix a sequence which satisfies  $\sum_{n=1}^{\infty} \varepsilon_n < 1$  (for instance  $\varepsilon_n = \frac{3}{\pi^2 n^2}$ ). The

above sequences are obtained by recurrence. In order to define the first elements, we consider a totally geodesic disk in  $\mathbb{H}^3$ . The choice of  $r_1$  is irrelevant.

Assume now we have defined  $\Sigma_n$  and  $r_n$  and satisfying items from (I<sub>n</sub>) to (IV<sub>n</sub>). We are going to construct the minimal surface  $\Sigma_{n+1}$ .

As the exhaustion  $\mathcal{S}$  is simple, then we know that  $S_{n+1} - \text{Int}(S_n)$  contains a unique nonannular component  $N$  which topologically is a pair of pants or an annulus with a handle. Label  $\gamma$  as the connected component of  $\partial N$  that is contained in  $\partial S_n$ . We label the connected components of  $\partial \Sigma_n$ ,  $\Gamma_1, \dots, \Gamma_k$ , in such a way that  $\gamma$  maps to  $\Gamma_k$  by the homeomorphism which maps  $S_n$  into  $\Sigma_n$ . Then, we apply Theorem 3.27 to  $\Sigma_n$  in the following way.

**Case 1.**  $N$  is a pair of pants.

The curve  $\Gamma_k$  bounds a disk  $D_k$  in  $\partial_\infty \mathbb{H}^3$  that does not intersect the other boundary curves of  $\Sigma_n$ . Consider an arc  $\Gamma \subset D_k$  so that  $\Gamma \cap \Gamma_k = \partial \Gamma$ . Then, we apply Theorem 3.27 to the configuration  $\Sigma_n \cup \Gamma$ . In this way, we construct a family  $\{T_m\}_{m \in \mathbb{N}}$  of properly embedded minimal surfaces obtained from  $\Sigma_n$  by adding a bridge  $B_m^1$  that “divides”  $\Gamma_k$  into two different curves in  $\partial_\infty \mathbb{H}^3$ . Note that the surfaces  $T_m$  have the same topology as  $S_{n+1}$ , for all  $m \in \mathbb{N}$ .

**Case 2.**  $N$  is a cylinder with a handle.

We construct the surface  $T_m$ , like in the previous case. But this time we add a second bridge  $B_m^2$  along a curve  $\sigma$  joining two opposite points in  $\partial B_m^1$  (see Figure 3). Notice that, in this way, the old annular component becomes an annulus with a handle. Again the resulting surfaces, that we still call  $T_m$ , are homeomorphic to  $S_{n+1}$ .

In both cases, we obtain a sequence of properly embedded, area minimizing surfaces,  $T_m$ , satisfying:

- (i)  $T_m$  is strictly  $L^\infty$ -stable and uniquely area minimizing.
- (ii)  $T_m$  admits a smooth extension to  $\bar{\mathbb{H}}^3$  and  $\bar{T}_m$  is diffeomorphic to  $S_{n+1}$ .
- (iii) The surfaces  $T_m \cap \overline{B(0, r_n)}$  are diffeomorphic to  $\Sigma_n \cap \overline{B(0, r_n)}$  and converge in the  $C^\infty$  topology to  $\Sigma_n \cap \overline{B(0, r_n)}$ , as  $\varepsilon \rightarrow 0$ .



Item (iii) and property (IV<sub>n</sub>) imply that  $T_m \cap \overline{B(0, r_i)}$  can be expressed as a normal graph over its projection  $\Sigma_{i,m} \subset \Sigma_i$ ,  $i = 1, \dots, n$ ;

$$T_m \cap \overline{B(0, r_i)} = \{\exp_p(h_{m,i}(p) \nu_i(p)) \mid p \in \Sigma_{i,m}\}.$$

Since, as  $m \rightarrow \infty$ , the surfaces  $T_m$  converge smoothly to  $\Sigma_n$  in  $B(0, r_n)$  and  $\Sigma_n$  satisfies (IV<sub>n</sub>), then we have:

$$(4.1) \quad \max\{|h_{m,i}|, |\nabla h_{m,i}|\} < \sum_{k=i+1}^{n+1} \varepsilon_k$$

for  $m$  large enough.

Then, we define  $\Sigma_{n+1} \stackrel{\text{def}}{=} T_m$ , where  $m$  is chosen sufficiently large in order to satisfy (4.1). We chose  $r_{n+1}$  big enough in order to guarantee that  $\Sigma_{n+1} \cap \overline{B(0, r_{n+1})}$  is diffeomorphic to  $S_{n+1}$ . It is clear that  $\Sigma_{n+1}$  so defined fulfills (I<sub>n+1</sub>),  $\dots$ , (IV<sub>n+1</sub>).

**Remark 4.2.** Taking into account the way in which we are using the bridge principle at infinity to modify the topology of  $\Sigma_n$ , it is important to notice that the new boundary curves of  $\Sigma_{n+1}$  are contained in the disk  $D_k \subset \partial_\infty \mathbb{H}^3$ .

Now, we have constructed our sequence of minimal surfaces  $\{\Sigma_n\}_{n \in \mathbb{N}}$ . Taking into account properties (IV<sub>n</sub>), for  $n \in \mathbb{N}$ , and using Ascoli-Arzelà's theorem, we deduce that the sequence of surfaces  $\{\Sigma_n\}_{n \in \mathbb{N}}$  converges to a properly embedded minimal surface  $\Sigma$  in the  $C^m$  topology, for all  $m \in \mathbb{N}$ . Moreover,  $\Sigma \cap \overline{B(0, r_i)}$  is a normal graph over its projection  $\Sigma_{i,\infty} \subset \Sigma_i$ , for all  $i \in \mathbb{N}$ , and the norm of the gradient of the graphing functions is at most 1 (see properties (IV<sub>n</sub>)).

Finally, we check that  $\Sigma$  satisfies all the statements in the theorem.

- $\Sigma$  is diffeomorphic to  $S$ . If we consider the (simple) exhaustions  $\{\Sigma \cap \overline{B(0, r_n)} \mid n \in \mathbb{N}\}$  of  $\Sigma$  and  $\{S_n \mid n \in \mathbb{N}\}$  of  $S$ , then we know that there exists a diffeomorphism  $\psi_n : S_n \rightarrow \Sigma \cap \overline{B(0, r_n)}$ . Furthermore, due to the way in which we have constructed  $\Sigma$ , we have that  $\psi_n|_{S_i} = \psi_i$ , for all  $i < n$ . Hence, we can construct a diffeomorphism  $\psi : S \rightarrow \Sigma$ .

If we consider on  $S$  the pull back of the metric of  $\Sigma$ , then  $\psi$  is the minimal embedding we are looking for.

- $\Sigma$  is area minimizing. The limit of area minimizing surfaces is area minimizing.

- The limit sets of distinct ends are disjoint. We are going to assume that  $\Sigma$  has at least two ends, otherwise this property does not make sense. Two different ends of  $\Sigma$ ,  $E_1$  and  $E_2$ , can be represented by two disjoint components,  $C_1$  and  $C_2$ , of  $\Sigma \setminus B(0, r_n)$ , for a sufficiently large  $n \in \mathbb{N}$ . Consider  $\partial_i = C_i \cap \overline{B(0, r_n)}$ ,  $i = 1, 2$ . Recall that  $\Sigma \cap \overline{B(0, r_n)}$  is a graph over  $\Sigma_n$ . Then, we label  $\partial_1^n$  and  $\partial_2^n$  the projection over  $\Sigma_n$  of  $\partial_1$  and  $\partial_2$ , respectively.

Observe that, from our method of construction,  $\partial_i$  (and  $\partial_i^n$ ) is a connected curve, for  $i = 1, 2$ . The curves  $\partial_1^n$  and  $\partial_2^n$  bound two different annular ends of  $\Sigma_n$  that we call  $A_1^n$  and  $A_2^n$ , respectively. Let  $\Gamma_i^n$  be the ideal boundary  $\partial_\infty A_i^n$ , for  $i = 1, 2$ . The curve  $\Gamma_i^n$  bounds a disk  $D_i^n \subset \partial_\infty \mathbb{H}^3$ ,  $i = 1, 2$ , and we know that  $D_1^n \cap D_2^n = \emptyset$ . Taking Remark 4.2 into account, we deduce that  $L(E_1) \subset D_1^n$  and  $L(E_2) \subset D_2^n$ . This concludes the proof.  $\square$

We would like to finish this section by pointing out that a suitable modification of the methods allow us to construct properly embedded area-minimizing surfaces so that the limit set is the whole ideal boundary  $\partial_\infty \mathbb{H}^3$ .

**Proposition 4.3.** *Let  $M$  be an open orientable surface. Then there exists a complete, proper, area-minimizing embedding  $f: M \rightarrow \mathbb{H}^3$  such that the limit set is  $\partial_\infty \mathbb{H}^3$ .*

*Proof.* We want to modify the proof of Theorem 4.1 as follows: we construct a sequence  $\{\Sigma'_n\}_{n \in \mathbb{N}}$  in such a way that it satisfies Properties (I<sub>n</sub>), ..., (IV<sub>n</sub>) (see page 16) and:

(V<sub>n</sub>) The Euclidean distance from  $\partial_\infty \Sigma_n$  to any point in  $\partial_\infty \mathbb{H}^3$  is less than  $1/n$ .

To do this, once we have obtained the minimal surface  $\Sigma_n$  satisfying (I<sub>n</sub>), ..., (IV<sub>n</sub>), then we proceed as follows: Let  $\Omega_1, \dots, \Omega_k$  be the connected components of  $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \Sigma_n$ . Take one of this components,  $\Omega_i$ ,  $i \in \{1, \dots, k\}$  and consider a totally geodesic disk  $D_i$  in  $\mathbb{H}^3$  satisfying:

- $D_i$  and  $\Sigma_n$  are disjoint;
- $\partial_\infty D_i \subset \Omega_i$ ;
- $\text{diam}_{\mathbb{H}^3}(\partial_\infty D_i) < \frac{1}{2n}$ ;
- $D_i \cup \Sigma_n$  is uniquely area minimizing (and strictly  $L^\infty$ -stable).

Let  $\Gamma_i$  be a smooth arch in  $\Omega_i$  which connects  $\partial_\infty \Sigma_n$  and  $\partial_\infty D_i$  and which is  $\frac{1}{2n}$  close to every point in  $\Omega_i$ . Then, we apply Theorem 3.27 to construct a new surface by connecting  $\Sigma_n$  with  $D_i$  by a bridge along the arc  $\Gamma_i$ . Notice that the surface obtained in this way has the topology as  $\Sigma_n$ . We call  $\Sigma'_n$  the surface obtained by repeating the above procedure for all  $i \in \{1, \dots, k\}$ . If the width of the bridges is sufficiently small we can guarantee that  $\Sigma_n$  satisfies (V<sub>n</sub>). So, the limit surface  $\Sigma$  would satisfy that its limit set  $L(\Sigma)$  is  $\partial_\infty \mathbb{H}^3$ .  $\square$

**4.1. Regularity of the boundary.** Although the minimal embedding constructed in Theorem 4.1 is limit of surfaces with smooth boundary, we cannot assert anything about the regularity at infinity of the minimal surface that we have obtained. In the case of finite topology, Oliveira and Soret [7] constructed minimal embeddings that extends smoothly to  $\bar{\mathbb{H}}^3$ . Hence, we shall center our attention on the case of open surfaces with infinite topology. If we do not care about the property that the limit sets of different ends were disjoint, then we can demonstrate the following:

**Theorem 4.4.** *Let  $S$  be an open surface with infinite topology, then there exists a proper area-minimizing embedding of  $S$  into  $\mathbb{H}^3$  such that the limit set in  $\partial_\infty \mathbb{H}^3$  is a smooth curve except for one point. Moreover the area-minimizing embedding extends smoothly to an embedding of  $S$  into  $\bar{\mathbb{H}}^3$  except for that point.*

*Proof.* Along the proof we will consider the upper half-space model of  $\mathbb{H}^3$ . So,  $\partial_\infty \mathbb{H}^3 = \{z = 0\} \cup \{\omega\}$ , where  $\omega$  represents the point of infinity in the compactification of the plane  $\{z = 0\}$ . Let  $\mathcal{S} = \{S_1 \subset S_2 \subset \dots \subset S_n \subset \dots\}$  a simple exhaustion for the surface  $S$ . For  $n \in \mathbb{N}$ , we define  $X_n = \{(x, y, z) \in \bar{\mathbb{H}}^3 : 2(n-1) < x < 2n-1\}$  and  $Y_n = \{(x, y, z) \in \bar{\mathbb{H}}^3 : 2n-1 < x < 2n\}$ .

Consider a totally geodesic disk  $D_n$  contained in the region  $X_n$  given by the semi-sphere centered at  $(2n-3/2, 0, 0)$  and radius  $r_n < 1/2$ . Let  $A_n$  the minimal annulus obtained by adding a bridge to  $D_n$  along a diameter of  $\partial_\infty D_n$ . Similarly,

we can construct a minimal disk with a handle  $T_n$ , included in the region  $Y_n$ . First we add a bridge at infinity  $B$  to a totally geodesic disk represented by a semi-sphere centered at  $(2n - 1/2, 0, 0)$  and radius  $r_n < 1/2$ . Later, we add a second bridge  $B'$  along a curve in  $\partial_\infty \mathbb{H}^3$  joining to opposites points of the ideal boundary of  $B$ . Notice that the surfaces  $A_n$  and  $T_n$ ,  $n \in \mathbb{N}$ , satisfy the hypothesis of our bridge principle at infinity (Theorem 3.27).

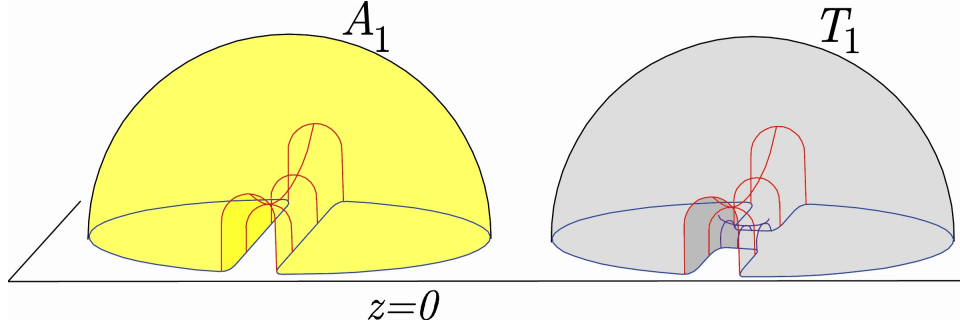


FIGURE 3. The surfaces  $A_1$  and  $T_1$

As in the proof of Theorem 4.1, we construct our surface inductively. The first element in our sequence is the totally geodesic disk  $\Sigma_1 = D_1$ . The second element in the sequence,  $\Sigma_2$ , is obtained by joining  $\Sigma_1$  with  $W_2 \in \{A_2, T_2\}$  by a bridge at infinity along a curve  $\Gamma_2$  with is contained in  $\partial_\infty \mathbb{H}^3 \cap \{x < 4\}$ . The choice of  $W_2$  depends on the topology of  $S_2 \setminus \text{Int}(S_1)$ . To add this bridge, we have to guarantee that  $\Sigma_1 \cup W_2$  satisfies the assumptions of Theorem 3.27. It is clear that  $\Sigma_1 \cup W_2$  is strictly  $L^\infty$ -stable, so we only need to check that it is *uniquely area-minimizing*. This can be guaranteed by applying a suitable homothetical shrinking to  $W_2$  with respect to  $(1/2, 0, 0)$  or  $(3/2, 0, 0)$  (depending on the nature of  $W_2$ ). Observe that the ideal boundary  $\partial_\infty \Sigma_2$  is a set of pairwise disjoint Jordan curves so that  $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \Sigma_2$  consists of a disjoint union of disks (actually, either one or two disks) and one unbounded connected component that is not simply connected and that we shall denote  $\mathcal{C}_2$ .

Assume that the surface  $\Sigma_n$  is constructed in such a way that  $\Sigma_n$  is diffeomorphic to  $S_n$  and  $\partial_\infty \Sigma_n$  consists of a finite set of pairwise disjoint Jordan curves and such that  $\partial_\infty \mathbb{H}^3 \setminus \partial_\infty \Sigma_n$  consists of disjoint union of disks joint with an connected component  $\mathcal{C}_n$  that is not simply connected ( $\mathcal{C}_n$  contains the point  $\omega$ .) We are going to show how to construct the surface  $\Sigma_{n+1}$ . We know that  $S_{n+1} \setminus \text{Int}(S_n)$  contains exactly one non-annular connected component that we call  $\Delta_{n+1}$ . Let  $\sigma_{n+1} \subset \partial_\infty \Sigma_n$  be the connected component of  $\partial_\infty \Sigma_n$  which corresponds to  $\partial \Delta_{n+1} \cap \partial S_n$  and let  $q_{n+1} = (x_{n+1}, y_{n+1}, z_{n+1})$  be the point of  $\sigma_{n+1}$  with the highest  $x$ -coordinate. We have that  $x_{n+1} \in [m, m+1]$  for some  $m \in \mathbb{N}$ .

Then we are going to construct a curve  $\Gamma_{n+1} \subset \mathcal{C}_n \cap \{m \leq x < 2(n+1)\}$  joining  $q_{n+1}$  and  $W_{n+1} \in \{A_{n+1}, T_{n+1}\}$ , where  $W_{n+1}$  depends on the topology of  $\Delta_{n+1}$ . To do this, we proceed as follows. The intersection of  $\{(t, 0, 0) : t \geq x_{n+1}\}$  and  $\overline{\mathcal{C}_n}$  consists of finite (disjoint) union of segments  $\alpha_1 \cup \dots \cup \alpha_l$  and a half-line  $r$ . Let  $\alpha_{l+1}$  be the piece of  $r$  joining  $\partial_\infty \Sigma_n$  and  $\partial_\infty W_{n+1}$ . For  $j \in \{1, \dots, l\}$ , label  $\beta_j$  the arc in  $\partial_\infty \Sigma_n$  that joins the end point of  $\alpha_j$  and the initial point of  $\alpha_{j+1}$ . Notice

that, from our method of construction, the  $x$ -coordinate is non-decreasing along  $\beta_j$ ,  $j = 1, 2, \dots, l$ . Let us define

$$\gamma = \alpha_1 * \beta_1 * \alpha_2 * \dots * \alpha_l * \beta_l * \alpha_{l+1}.$$

The curve  $\Gamma_{n+1}$  is a suitable perturbation of  $\gamma$  satisfying that  $\Gamma_{n+1} \subset \mathcal{C}_n \cap \{m \leq x < 2(n+1)\}$ .

Again, up to a suitable shrinking of  $W_{n+1}$  we can assume that we are in the conditions for applying Theorem 3.27, and so, we obtain  $\Sigma_{n+1}$  by adding a bridge along  $\Gamma_{n+1}$  to  $\Sigma_n \cup W_{n+1}$ . Observe that, up to a infinitesimal perturbation of the bridge boundary, we can assume that  $\partial_\infty \Sigma_{n+1}$  meets the  $x$ -axis transversally.

It is important to notice that the sequence of surfaces  $\{\Sigma_n\}_{n \in \mathbb{N}}$  constructed in this way satisfies that, for all  $r > 0$ , the ideal boundary  $\partial_\infty \Sigma_n$  intersects the region  $\{x \leq r\}$  in the same set of arcs, for  $n$  sufficiently large.

Reasoning as in the proof Theorem 4.1, we can guarantee that the sequence  $\{\Sigma_n\}_{n \in \mathbb{N}}$  converges smoothly on compact sets to a properly embedded minimal surface  $\Sigma$ . From the previous observation, it is clear that  $\partial_\infty \Sigma \cap \{x \leq r\} = \partial_\infty \Sigma_n \cap \{x \leq r\}$ , for  $n \in \mathbb{N}$  large enough. Thus,  $\partial_\infty \Sigma$  is smooth except for the point of infinity  $\omega \in \partial_\infty \mathbb{H}^3$ .  $\square$

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